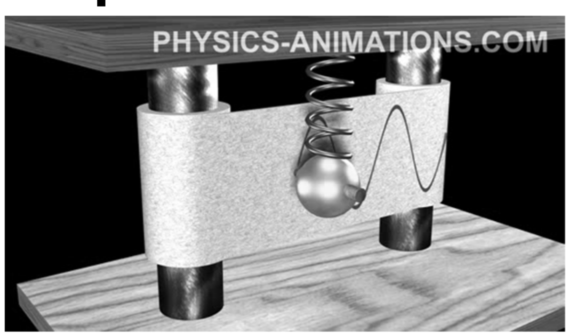
Simple Harmonic Motion
Springs - Part 2

1

Displacement in SHM

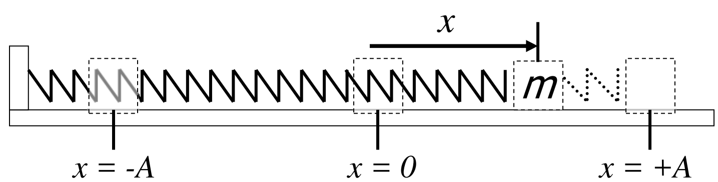


- As seen in our lab, when a mass hanging from a spring is pulled away from its equilibrium position, it will be put into simple harmonic motion.

2

Displacement in SHM

- The same motion can be observed in a mass attached to a horizontal spring oscillating on a frictionless surface.



- Displacement is positive when the position is to the right of the equilibrium position (x = 0) and negative when located to the left.
- The maximum displacement is called the amplitude A.

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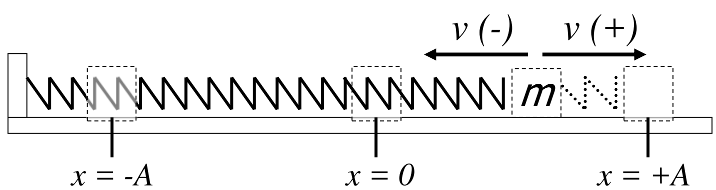
Velocity in SHM



- Velocity is positive when moving to the right and negative when moving to the left.
- It is zero at the end points and a maximum at the midpoint in either direction (+ or -).

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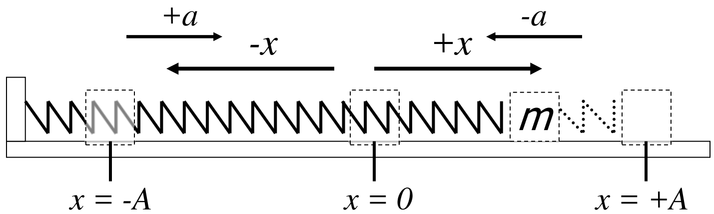
Acceleration in SHM



- Acceleration is in the direction of the restoring force. (*a* is positive when x is negative, and negative when x is positive.)

$$F = ma = -kx$$

- Acceleration is a maximum at the end points and it is zero at the center of oscillation.

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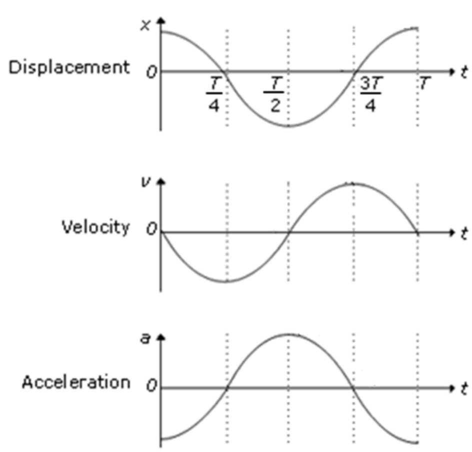
Comparing Displacement, Velocity and Acceleration of SHM



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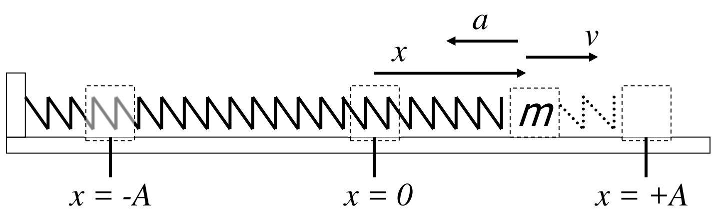
Acceleration vs. Displacement



Given the spring constant, the displacement, and the mass, the acceleration can be found from:

$$F = ma = -kx \quad \text{or} \quad a = \frac{-kx}{m}$$

Note: Acceleration is always opposite to displacement.

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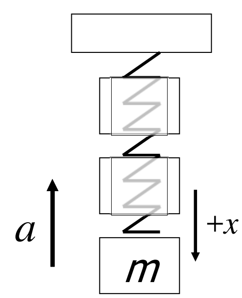
Example 1: A 2-kg mass hangs at the end of a spring whose constant is k = 400 N/m. The mass is displaced a distance of 12 cm and released. What is the acceleration at the instant the displacement is x = +7 cm?

$$a = \frac{-kx}{m}$$

$$a = \frac{(400\ \text{N/m})(+7.0\text{m})}{2\ \text{kg}}$$

$$\boldsymbol{a} = -14.0\ \text{m/s}^2$$



Note: When the displacement is +7 cm (downward), the acceleration is -14.0 m/s² (upward) independent of motion direction.

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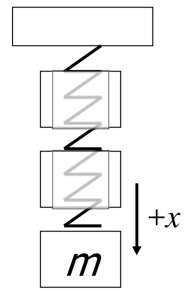
Example 2: What is the maximum acceleration for the 2-kg mass in the previous problem? (A = 12 cm, k = 400 N/m)

The maximum acceleration occurs when the restoring force is a maximum; i.e., when the stretch or compression of the spring is largest.



$$F = ma = -kx \qquad x_{max} = \pm A$$

$$a_{max} = \frac{-kA}{m} = \frac{-400\ \text{N}(\pm 0.12\ \text{m})}{2\ \text{kg}}$$

Maximum Acceleration: $a_{max} = \pm\ 24.0\ \text{m/s}^2$

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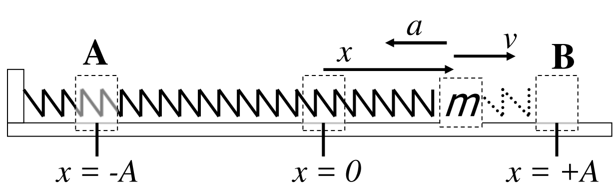
Energy of a Vibrating System:



- At points A and B, the velocity is zero and the acceleration is a maximum. The total energy is:

$$U + K = ½kA^2 \quad x = \pm A \text{ and } v = 0.$$

- At any other point: $U + K = ½mv^2 + ½kx^2$

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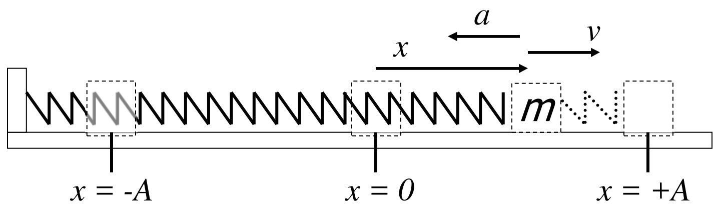
Conservation of Energy

The total mechanical energy (U + K) of a vibrating system is constant; i.e., it is the same at any point in the oscillating path.



For any two points A and B, we may write:

$$K_A + U_A = K_B + U_B$$

$$\tfrac{1}{2}mv_A^2 + \tfrac{1}{2}kx_A^{\ 2} = \tfrac{1}{2}mv_B^2 + \tfrac{1}{2}kx_B^{\ 2}$$

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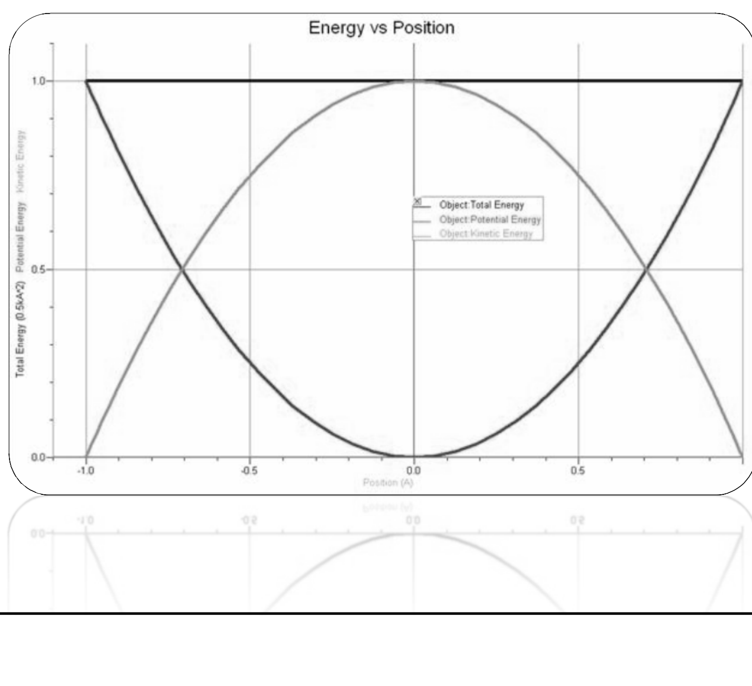
Energy in SHM



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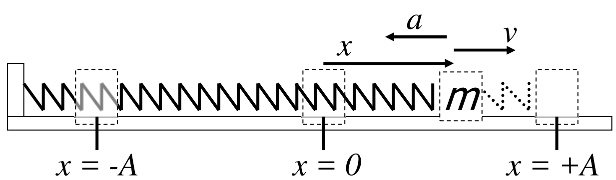
Velocity as Function of Position



$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 \qquad v = \sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$

v_{max} when x = 0: $$v_{\max} = \sqrt{\frac{k}{m}}A$$

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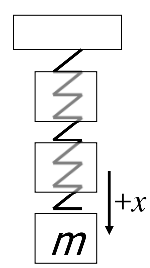
Example 3: A 2-kg mass hangs at the end of a spring whose constant is k = 800 N/m. The mass is displaced a distance of 10 cm and released. What is the velocity at the instant the displacement is x = +6 cm?



$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

$$0 + \tfrac{1}{2}(800)(0.1)^2 = \tfrac{1}{2}(2)(v)^2 + \tfrac{1}{2}(800)(.06)^2$$

$$0 + 4 = v^2 + 1.44$$

$$v^2 = 2.56$$

$$v = 1.60 \text{ m/s}$$

$$v = \sqrt{\frac{k}{\text{m}}}\sqrt{\text{A}^2 - \text{x}^2}$$

$$v = \sqrt{\frac{800 \text{ N/m}}{2 \text{ kg}}}\sqrt{(0.1)^2 - (0.06 \text{ m})^2}$$

$$v = \pm 1.60 \text{ m/s}$$

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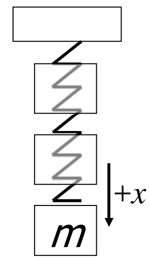
Example 3 (Cont.): What is the maximum velocity for the previous problem?
(A = 10 cm, k = 800 N/m, m = 2 kg.)

The velocity is maximum when x = 0:



$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

$$0 + \tfrac{1}{2}(800)(0.1)^2 = \tfrac{1}{2}(2)v^2 + 0$$

$$4 = v^2$$

$$v = 2\ m/s$$

$$v = \sqrt{\frac{k}{m}}A = \sqrt{\frac{800\ \text{N/m}}{2\ \text{kg}}}(0.1\ \text{m})$$

$$\boxed{v = \pm\, 2.00\ \text{m/s}}$$

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The Period and Frequency as a Function of *a* and *x*.

- From our lab we found that the quadratic relationship between period (T) and mass (m)

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

- From Newton's 2nd Law and Hooke's Law we found:

$$\boxed{a = \frac{-kx}{m}} \Rightarrow \frac{a}{x} = -\frac{k}{m} \Rightarrow \frac{m}{k} = -\frac{x}{a}$$

Substituting we find…

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The Period and Frequency as a Function of a and *x*.

For any body undergoing simple harmonic motion:

$$T = 2\pi\sqrt{\frac{-x}{a}}$$

And since $f = \frac{1}{T}$ $\Rightarrow$ $f = \frac{1}{2\pi}\sqrt{\frac{-a}{x}}$

The frequency and the period can be found if the displacement and acceleration are known. Note that the signs of a and x will always be opposite.

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Acceleration as a Function of f and *x*.

- Rewriting the previous equation we get:

$$f = \frac{1}{2\pi}\sqrt{\frac{-a}{x}} \Rightarrow -a = 4\pi^2 f^2 x$$

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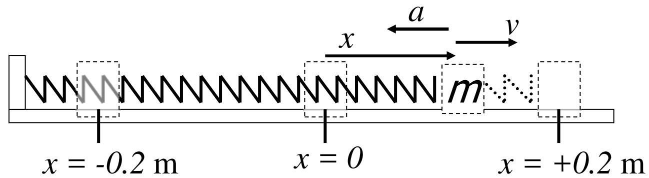
Example 4: **The frictionless system shown below has a 2-kg mass attached to a spring (k = 400 N/m). The mass is displaced a distance of 20 cm to the right and released. What is the frequency of the motion?**



$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{400\text{ N/m}}{2\text{ kg}}}$$

$$\boxed{f = 2.25\text{ Hz}}$$

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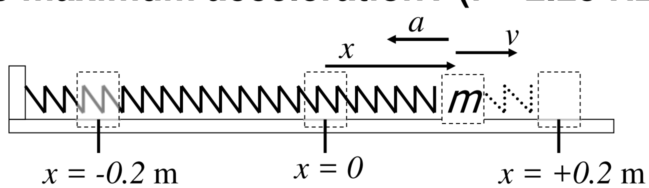
Example 4 (Cont.): **Suppose the 2-kg mass of the previous problem is displaced 20 cm and released (k = 400 N/m). What is the maximum acceleration? (*f* = 2.25 Hz*)***



Acceleration is a maximum when $x = \pm A$

$$a = -4\pi^2 f^2 x = -4\pi^2 (2.25\text{ Hz})^2 (\pm 0.2\text{ m})$$

$$\boxed{a = \pm 40\text{ m/s}^2}$$

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